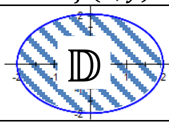
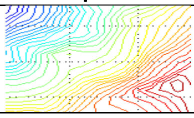
MEHRDIMENSIONALE ANALYSIS

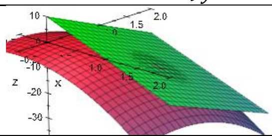
Funktionen von mehreren Variablen

Allgemein	$z = f(x,y)$	$f: \mathbb{R}^2 \to \mathbb{R}$ ordnet jedem Zahlenpaar (x,y) einen Werz Z zu.
Definitionsmenge **Definitionsbereich**		$\mathbb{D} = \{(x,y) \; \vert x^2 + y^2 \leq 4\}$ $\mathbb{D} = \{(x,y) \; \vert z \in \mathbb{R}^2\} = \{(x,y) \; \vert x \in \mathbb{R}, y \in \mathbb{R}\}$
Formen	**implizit**	$5x + 2y = 5$
	explizit	$y = 5 - 5x$ (nach Variablen aufgelöst)
Niveau-Höhenlinien **Isolinien**		Schnitt mit verschiedenen z-Ebenen $f(x,y) = \cdots$ $k = \cdots$

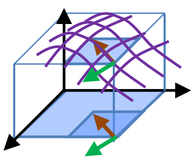
Ebene	**Grundform**	**Achsenabschnitte**	**Keine Funktion aber Ebene**
	$z = ax + by + c$	$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} = 1$	$x = 4$

Partielle Ableitung

Partielle Ableitungen	Ableitung nach x / x-Achse	Ableitung nach y / y-Achse	2te Ableitung nach x/x-Achse	2te Ableitung nach y/y-Achse	gemischte partielle Ableitungen
	$\frac{\partial f}{\partial x} = f_x$	$\frac{\partial f}{\partial y} = f_y$	$\frac{\partial^2 f}{\partial x^2} = f_{xx}$	$\frac{\partial^2 f}{\partial y^2} = f_{yy}$	$\frac{\partial^2 f}{\partial x \, \partial y} = f_{xy} = f_{yx}$

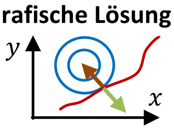
Gradient senkrecht auf Isolinie	$\vec{\nabla} f(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$	Richtung = Richtung des steilsten Anstiegs Betrag = Maximaler Aufstieg
Tangentialebene		Funktionswerte / Ableitung in x / Ableitung in y identisch: $z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

Anwendungen der partiellen Ableitungen

Extremalstellen	$\vec{\nabla} f(x,y) = 0 = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$	Suchen von Maximum, Minimum oder Sattelpunkten Unterteilen in Funktion, auf Strecke und an Endpunkten
Geometrie der 2ten Ableitung	 $f(x,y)$ $\vec{\nabla} f$ $\vec{v}$	(siehe Tabelle unten)
Hessematrix H und Taylor Reihe	$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$	$f(\vec{x_0} + \vec{\Delta x}) = f(\vec{x_0}) + \vec{\nabla} f(\vec{x_0}) * \vec{\Delta x} + \frac{1}{2} \vec{\Delta x}^T \, H \, \vec{\Delta x}$
Richtungsableitung	$\frac{\partial f}{\partial \vec{v}}(x_0,y_0) = \frac{\vec{v}}{\lvert\vec{v}\rvert} * \vec{\nabla} f(x_0,y_0)$	Steigung in Richtung $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
	$\frac{\partial f}{\partial \vec{v}} = 0, \quad \vec{\nabla} f * \vec{v} = 0$	Bewegung auf Höhenlinie (da Gradient ⊥ Höhenlinie)
Differentiation auf Pfaden	$z(t) = f\big(x(t), y(t)\big)$	Höhenprofil entlang des Pfades $z(x,y)$
	$Var\ 1: \frac{dz}{dt} = \frac{d}{dt} f\big(x(t), y(t)\big)$ $Var\ 2: \frac{dz}{dt} = f_x * \dot{x} + f_y * \dot{y}$	Steigung entlang eines Pfades $z(x,y)$ „Verallgemeinerte Kettenregel“

Geometrie der 2ten Ableitung:

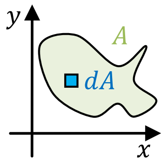
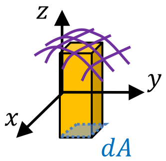
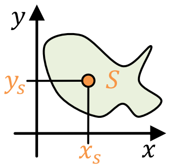
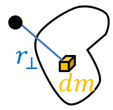
$\vec{\nabla} f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$		$\Delta(x,y) = f_{xx} * f_{yy} - f_{xy}^2$		
		< 0	$= 0$	> 0
f_{xx} f_{yy}	< 0	Sattelpunkt	part. Abl. höherer Ordnung	lok. Max.
	$= 0$			
	> 0			lok. Min.

Extremwerte mit Nebenbedingungen NB	**Grafische Lösung** 	**Lagrange Funktion** Unklar ob Min oder Max, da 1ter Ordnung	$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$ $L_x = 0 \quad L_y = 0 \quad L_\lambda = 0$
	Höhenlinien $\vec{\nabla} f \parallel \vec{\nabla} g$	**Alternativ durch Einsetzen** NB(explizit) in Funktion setzen	$\bar{f}(x,y) = f\big(x, y(x)\big) = \cdots$ $f_x = 0, \quad f_y = 0$

Max/Min	$f(x,y)$
NB	$g(x,y) = 0$

Totales Differential =Talyor-Entwicklung in 2D (1. Ordnung)	$z_0 + \Delta z = f(x_0 + \Delta x, y_0 + \Delta y)$	**exakt**
	$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$	**Näherung durch Tangentialebene** auch mit mehr als 2 part. Ableitungen Fehler: $\sigma = \lvert \Delta z_{exakt} - \Delta z_{Diff} \rvert$

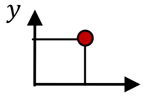
Zweifachintegrale

Flächenberechnung: Ebene	Flächenberechnung: Oberfläche O_b	Volumen	Schwerpunkt S: homogen	Schwerpunkt S: inhomogen	Trägheitsmoment J_z
$A = \int_A 1\, dA$	$= \int_A \sqrt{{f_x}^2 + {f_y}^2 + 1}\; dA$	$V = \int_A f(x,y)\, dA$	$x_s = \frac{1}{A}\int_A x\, dA$ $y_s = \frac{1}{A}\int_A y\, dA$	$x_s = \frac{1}{M}\int_A x\,\rho(x,y)\, dA$ $y_s = \frac{1}{M}\int_A y\,\rho(x,y)\, dA$	$= \delta \int_A (x^2+y^2)\, dA$
				$Dichte: [\rho] = \frac{kg}{m^2}$ $M = \int_A \rho(x,y)\, dA$	

Achten, ob Integral im Positiven bleibt!!

Integrale von Innen nach Aussen lösen

Kartesische Koordinaten



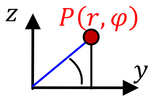
$= \Delta x * \Delta y$

dy dx

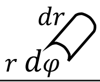
$r = \sqrt{x^2+y^2}$
$\tan\varphi = {}^{y}/_{x}$

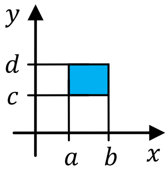
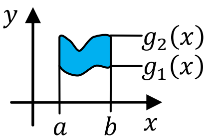
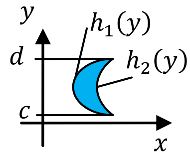
$x = r * \cos\varphi$
$y = r * \sin\varphi$

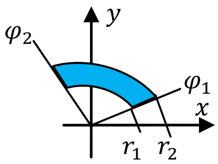
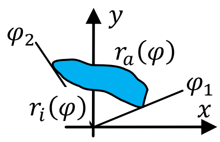
Polar Koordinaten



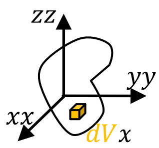
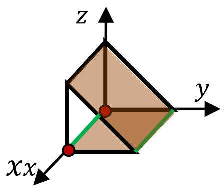
$\Delta A = r * dr * d\varphi$



Allgemein	$y = g(x)$	$x = h(y)$
$V = \int_a^b \int_c^d f(x,y)\; dy\; dx$	$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\; dy\; dx$	$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\; dx\; dy$
		
vertauschbar	nicht vertauschbar	nicht vertauschbar

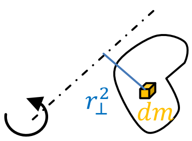
$V = \int_{\varphi_1}^{\varphi_2} \int_{r_1}^{r_2} f(r,\varphi)\, r\; dr\; d\varphi$	$V = \int_{\varphi_1}^{\varphi_2} \int_{r_i(\varphi)}^{r_2(\varphi)} f(r,\varphi)\, r\; dr\; d\varphi$
	
vertauschbar	nicht vertauschbar

Dreifachintegrale

Allgemein	Ausrechnen von Dreifachintegralen			
$V = \int_V f(x,y,z)\; dV$	$V = \int_0^1 \int_0^1 \int_0^{1-y} 1\; dz\; dy\; dx$			
$f\colon \mathbb{R}^3 \to \mathbb{R}$ 		Punkte	$x = 0$	$x = 1$
		Linien	$y = 0$	$y = 1$
		Flächen	$z = 0$	$z = 1 - y$

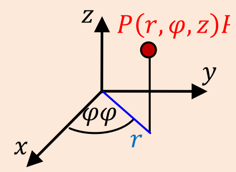
Volumen	$V = \int_V 1\; dV$
homogene Masse	$M = \int_V \rho(x,y,z)\; dV$
potenzielle Energie	$E = \int_V dE = \delta\, g \int_V z\; dV$

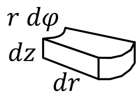
Grenzen dürfen nur Variablen äusserer Integrale besitzen.

Massenträgheitsmoment: allgemein	Massenträgheitsmoment: homogener	Massenträgheitsmoment: inhomogen	Schwerpunkt: homogen	Schwerpunkt: inhomogen
$J = m * r_\perp^2 = \int_V dJ\; dV$	$J = \delta \int_V r_\perp^2\; dV$	$J = \int_V r_\perp^2\, \delta(x,y,z)\; dV$	$x_s = \frac{1}{V}\int_V x\; dV$	$x_s = \frac{1}{M}\int_V \rho(x,y,z)\, x\; dV$
	$J_z = \delta \int_V (x^2+y^2)\; dV$	$J_z = \int_V (x^2+y^2)\, \delta(x,y,z)\; dV$	$y_s = \frac{1}{V}\int_V y\; dV$	$y_s = \frac{1}{M}\int_V \rho(x,y,z)\, y\; dV$
	$J_x = \delta \int_V (y^2+z^2)\; dV$	$J_x = \int_V (y^2+z^2)\, \delta(x,y,z)\; dV$	$z_s = \frac{1}{V}\int_V z\; dV$	$z_s = \frac{1}{M}\int_V \rho(x,y,z)\, z\; dV$
	$J_y = \delta \int_V (x^2+z^2)\; dV$	$J_y = \int_V (x^2+z^2)\, \delta(x,y,z)\; dV$		

Koordinatensysteme

Zylinderkoordinaten





$\boldsymbol{dV}$:
$r\,dr\,d\varphi\,dz$

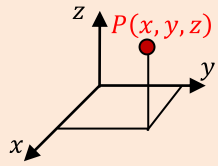
Zylinder → Kartesisch:

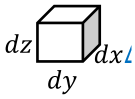
$x = r * \cos\varphi$
$y = r * \sin\varphi$
$z = z$

Kartesisch → Zylinder:

$r = \sqrt{x^2 + y^2}$
$\tan\varphi = \frac{y}{x}$
$z = z$

Kartesische Koordinaten





$\boldsymbol{dV}$:
$dx\,dy\,dz$

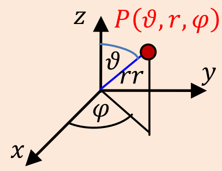
Kartesisch → Kugel:

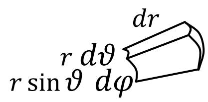
$r = \sqrt{x^2 + y^2 + z^2}$
$\tan\varphi = {}^{y}/_{x}$
$\cos\vartheta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

Kugel → Kartesisch:

$x = r * \cos\varphi * \sin\vartheta$
$y = r * \sin\varphi * \sin\vartheta$
$z = r * \cos\vartheta$

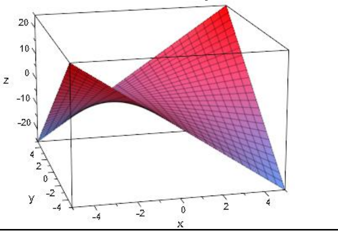
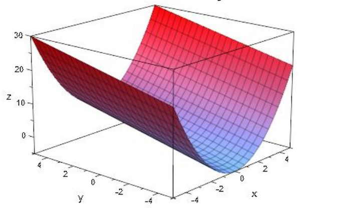
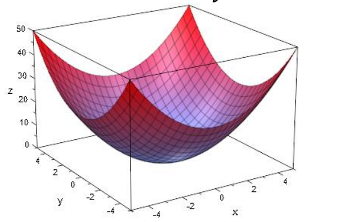
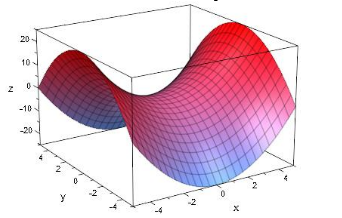
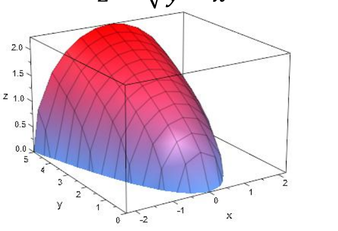
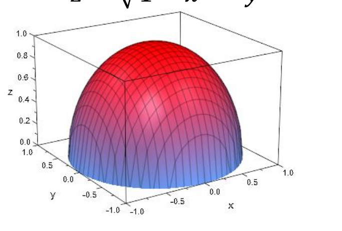
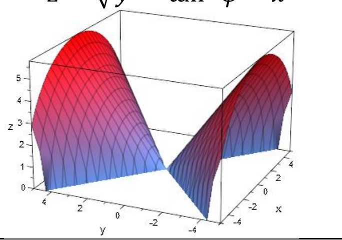
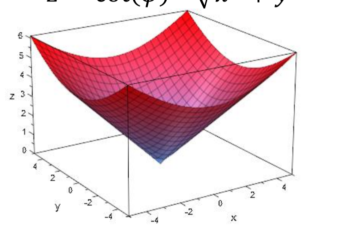
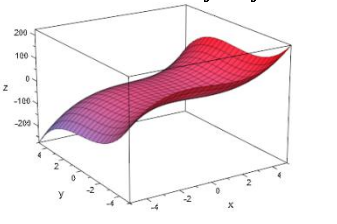
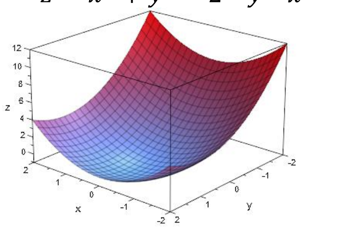
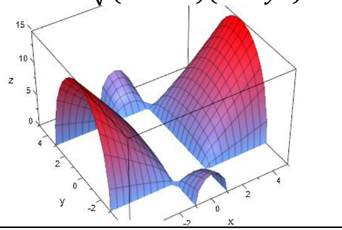
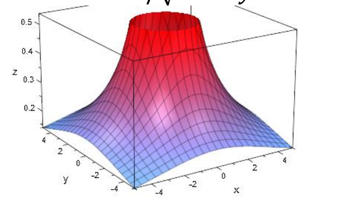
Kugelkoordinaten





$\boldsymbol{dV}$:
$r^2 \sin\vartheta\,dr\,d\varphi\,d\vartheta$
$0 \le \vartheta \le \pi$
$0 \le \varphi \le 2\pi$

Beispiele von Funktionen

$z = x * y$ 	$z = x^2 + y$ 	$z = x^2 + y^2$ 	$z = x^2 - y^2$ 
$z = \sqrt{y - x^2}$ 	$z = \sqrt{1 - x^2 - y^2}$ 	$z = \sqrt{y^2 * \tan^2\varphi - x^2}$ 	$z = \cot(\varphi) * \sqrt{x^2 + y^2}$ 
$z = x^3 + x * y - y^3$ 	$z = x^2 + y^2 - 2 * y * x$ 	$z = \sqrt{(x^2 - 1)(9 - y^2)}$ 	$z = 1/\sqrt{x^2 + y^2}$ 

Wichtige Funktionen

Kugel	Einheitskugel	Kreis (kartesisch)	Kreis (polar)	Einheitskreis	Ellipse
$R^2 = x^2 + y^2 + z^2$	$1 = x^2 + y^2 + z^2$	$R^2 = x^2 + y^2$	$x(t) = R * \cos(t)$ $y(t) = R * \sin(t)$ $\quad 0 \le t \le 2\pi$	$1 = x^2 + y^2$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$